JOURNAL OF COMPUTATIONAL PHYSICS **72**, 501–503 (1987)

Note

Split-Step Spectral Method for Nonlinear Schrödinger Equation with Absorbing Boundaries

By application of spectral methods [1] the computational solution of nonlinear partial differential equations has been improved in accuracy as well as efficiency in particular on vector computers. Fourier spectral methods [2] require periodic boundary conditions often in contrast to the actual physical problems where modelling by outflow boundary conditions may be appropriate in many cases.

In this note we consider the cubic nonlinear Schrödinger equation (NLS) which occurs in nonlinear optics [3], deep water wave theory [4], plasma physics [5], biomolecular dynamics [6], e.g. The equation can be solved numerically by the split-step Fourier method (SSFM) described in [7, 8]. We generalize the method by including an additional term in the partial differential equation with the effect of absorbing outgoing radiation at the boundaries. The applications of SSFM requires periodic boundary conditions. However, the drawback of these conditions is eliminated by our new method.

The NLS with periodic boundary conditions is given by

$$iu_t + \tfrac{1}{2}u_{xx} + |u|^2\, u = 0, \tag{1a}$$

$$u(-L/2, t) = u(L/2, t), \qquad \text{and} \qquad u_x(-L/2, t) = u_x(L/2, t) \tag{1b}$$

$-L/2 \leqslant x \leqslant L/2$, $-\infty < t < \infty$, and $u = u(x, t)$.

The SSFM in its original form consists of two steps. First, the nonlinear part of Eq. (1a), $iu_t + |u|^2\, u = 0$, is solved by means of the simple wave solution $u(x, t) = u(x, 0) \exp(i|u(x, 0)|^2\, t)$. Second, the linear part of Eq. (1a), $iu_t + \frac{1}{2}u_{xx} = 0$, is solved by means of Fourier transformation.

Our modified verion of NLS is

$$iu_t + \tfrac{1}{2}u_{xx} + |u|^2\, u + i\gamma(x)\, u = 0, \tag{2a}$$

$$u(-L/2, t) = u(L/2, t), \qquad \text{and} \qquad u_x(-L/2, t) = u_x(L/2, t), \tag{2b}$$

where the real function $\gamma(x)$ in the absorbing term, $i\gamma(x)\, u$, is given by

$$\gamma(x) = \gamma_0(\operatorname{sech}^2[\alpha(x - L/2)] + \operatorname{sech}^2[\alpha(x + L/2)]). \tag{2c}$$

As seen in Fig. 1 we introduce smooth losses at the boundaries $x = -L/2$ and $x = L/2$ through this choice of γ.

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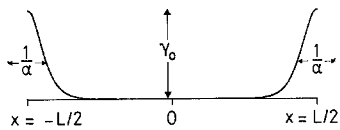


FIG. 1. The absorption function $\gamma(x)$ (2c) introduces losses in the neighborhood of the periodic boundaries at $x = \pm L/2$. Parameters γ_0 and α in (2c) must be chosen such that the scattering from the "absorption walls," $\operatorname{sech}^2[\alpha(x \mp L/2)]$, is small.

In the corresponding new generalized split-step method we first solve the nonlinear part of Eq. (2a)

$$i\tilde{u}_t + |\tilde{u}|^2\,\tilde{u} + i\gamma(x)\,\tilde{u} = 0 \tag{3}$$

for which we have found the exact solution

$$\tilde{u}(x, t) = \tilde{u}(x, 0)\exp\{i\,|\tilde{u}(x, 0)|^2\,(1 - e^{-2\gamma t})/2\gamma - \gamma t\} \tag{4}$$

by inspection. Second, the linear part, $i\tilde{\tilde{u}}_t + \frac{1}{2}\tilde{\tilde{u}}_{xx} = 0$, is solved in Fourier space by

$$\tilde{\tilde{U}}(k, t) = \tilde{\tilde{U}}(k, 0)\,\exp\{-ik^2 t/2\}. \tag{5}$$

Also in our generalized SSFM the solution is advanced one time step Δt by (i) obtaining $\tilde{u}(x, \Delta t)$ from $u(x, 0)$ by means of (4) with $\tilde{u}(x, 0) = u(x, 0)$, (ii) inserting the Fourier transform of $\tilde{u}(x, \Delta t)$ as $\tilde{\tilde{U}}(k, 0)$ in (5)

$$\tilde{\tilde{U}}(k, \Delta t) = \int_{-\infty}^{\infty} \tilde{u}(x, \Delta t)\exp\{ikx\}\,dx\,\exp\{-ik^2\,\Delta t/2\}, \tag{6}$$

and (iii) transforming the resulting $\tilde{\tilde{U}}(k, \Delta t)$ back to x-space

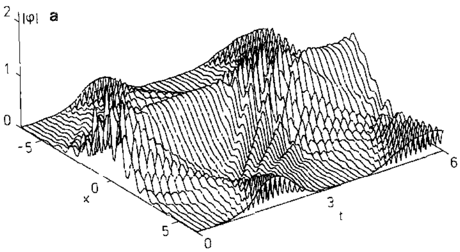
$$u(x, \Delta t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{U}}(k, \Delta t)\,\exp\{-ikx\}\,dk. \tag{7}$$

This method is second-order accurate in Δt and all orders in Δx and is unconditionally stable according to linear analysis [8].

Figure 2 shows the time development of the initial condition

$$u(x, 0) = (1 + 0.6\cos 7x)\operatorname{sech} x \tag{8}$$

in two cases: (a) subjected to the classical NLS dynamics given by (1) and (b) subjected to the NLS dynamics with absorption given by (2). The initial condition (8) desribes an NLS 1-soliton [9] with radiation superposed. In case (a) the radiation cannot escape from the system owing to the spatial periodicity and eventually destroys the 1-soliton. In case (b) the radiation is essentially absorbed already at the first passage of the boundary leaving the 1-soliton undisturbed.



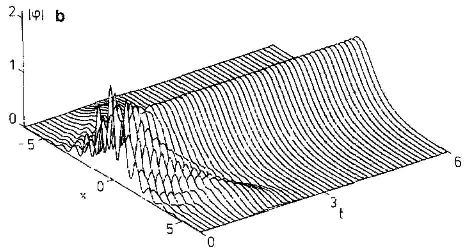


FIG. 2. Evolution of NLS 1-soliton (8) with dynamics given by (a) classical NLS (1), (b) NLS with absorption (2). Parameters $L = 12.8$, $\gamma_0 = 20$, and $\alpha = 1$. Resolution $\Delta x = 0.1$ and $\Delta t = 0.005$.

The difference between Figs. 2a and 2b demonstrates the importance of adding absorption in the NLS equation. This new trick makes the SSFM much more applicable to the physical problems mentioned in the introduction.

ACKNOWLEDGMENTS

The financial support of the Danish Council for Scientific and Industrial Research and of the European Research Office of the United States Army through Contract DAJA-45-85-C-0043 is acknowledged.

REFERENCES

1. D. GOTTLIEB AND S. A. ORSZAG, *Numerical Analysis of Spectral Methods: Theory and Applications*, CBMS-NSF Regional Conf. Ser. in Appl. Math. Vol. 26 (Soc. Indus. Appl. Math., Philadelphia, 1977), p. 1.
2. D. GOTTLIEB, M. Y. HUSSAINI, AND S. A. ORSZAG, in *Proceedings, Symposium on Spectral Methods for PDE*, edited by R. G. Voigt, D. Gottlieb, and M. Y. Hussaini, SIAM Monograph, Soc. Indus. Appl. Math., Philadelphia, 1984), p. 1.
3. A. HASEGAWA AND F. TAPPERT, *Appl. Phys. Lett.* **23**, 142 (1973).
4. D. J. BENNEY AND A. C. NEWELL, *J. Math. Phys.* **46**, 133 (1967).
5. G. J. MORALES AND Y. C. LEE, preprint, UCLA, 1975 (unpublished).
6. A. S DAVYDOV, *Phys. Scripta* **20**, 387 (1979).
7. T. R. TAHA AND M. J. ABLOWITZ, *J. Comput. Phys.* **55**, 203 (1984).
8. R. H. HARDIN AND F. D. TAPPERT, *SIAM Rev. Chronicle* **15**, 423 (1973).
9. J. SATSUMA AND N. YAJIMA, *Prog. Theor. Phys. Supp.* **55**, 284 (1974).

RECEIVED: June 24, 1986; REVISED: November 20, 1986

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